

# ON THE INDEX-CONJECTURE OF LENGTH FOUR MINIMAL ZERO-SUM SEQUENCES II

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**ABSTRACT.** Let  $G$  be a finite cyclic group. Every sequence  $S$  over  $G$  can be written in the form  $S = (n_1g) \cdots (n_lg)$  where  $g \in G$  and  $n_1, \dots, n_l \in [1, \text{ord}(g)]$ , and the index  $\text{ind}S$  of  $S$  is defined to be the minimum of  $(n_1 + \cdots + n_l)/\text{ord}(g)$  over all possible  $g \in G$  such that  $\langle g \rangle = G$ . A conjecture says that if  $G$  is finite such that  $\gcd(|G|, 6) = 1$ , then  $\text{ind}(S) = 1$  for every minimal zero-sum sequence  $S$ . In this paper, we prove that the conjecture holds if  $S$  is reduced and the (A1) condition is satisfied (see [19]).

*Key Words:* cyclic group, minimal zero-sum sequence, index of sequences, reduced.

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## 1. INTRODUCTION

Throughout the paper, let  $G$  be an additively written finite cyclic group of order  $|G| = n$ . By a sequence over  $G$  we mean a finite sequence of terms from  $G$  which is unordered and repetition of terms is allowed. We view sequences over  $G$  as elements of the free abelian monoid  $\mathcal{F}(G)$  and use multiplicative notation. Thus a sequence  $S$  of length  $|S| = k$  is written in the form  $S = (n_1g) \cdots (n_kg)$ , where  $n_1, \dots, n_k \in \mathbb{N}$  and  $g \in G$ . We call  $S$  a *zero-sum sequence* if  $\sum_{j=1}^k n_jg = 0$ . If  $S$  is a zero-sum sequence, but no proper nontrivial subsequence of  $S$  has sum zero, then  $S$  is called a *minimal zero-sum sequence*. Recall that the index of a sequence  $S$  over  $G$  is defined as follows.

**Definition 1.1.** For a sequence over  $G$

$$S = (n_1g) \cdots (n_kg), \quad \text{where } 1 \leq n_1, \dots, n_k \leq n,$$

the index of  $S$  is defined by  $\text{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } \langle g \rangle = G\}$ , where

$$(1.1) \quad \|S\|_g = \frac{n_1 + \cdots + n_k}{\text{ord}(g)}.$$

Clearly,  $S$  has sum zero if and only if  $\text{ind}(S)$  is an integer.

**Conjecture 1.2.** Let  $G$  be a finite cyclic group such that  $\gcd(|G|, 6) = 1$ . Then every minimal zero-sum sequence  $S$  over  $G$  of length  $|S| = 4$  has  $\text{ind}(S) = 1$ .

The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. It was first addressed by Kleitman-Lemke (in the conjecture [9, page 344]), used as a key tool by Geroldinger ([6, page 736]), and then

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investigated by Gao [3] in a systematical way. Since then it has received a great deal of attention (see for example [1, 2, 4, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18]). A main focus of the investigation of index is to determine minimal zero-sum sequences of index 1. If  $S$  is a minimal zero-sum sequence of length  $|S|$  such that  $|S| \leq 3$  or  $|S| \geq \lfloor \frac{n}{2} \rfloor + 2$ , then  $\text{ind}(S) = 1$  (see [1, 14, 16]). In contrast to that, it was shown that for each  $k$  with  $5 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1$ , there is a minimal zero-sum subsequence  $T$  of length  $|T| = k$  with  $\text{ind}(T) \geq 2$  ([13, 15]) and that the same is true for  $k = 4$  and  $\gcd(n, 6) \neq 1$  ([13]). The left case leads to the above conjecture.

In [12], it was proved that Conjecture 1.2 holds true if  $n$  is a prime power. In [11], it was proved that Conjecture 1.2 holds for  $n = p_1^\alpha \cdot p_2^\beta$ , ( $p_1 \neq p_2$ ), and at least one  $n_i$  co-prime to  $|G|$ .

In [19], it was proved that Conjecture 1.2 holds if the sequence  $S$  is reduced and at least one  $n_i$  co-prime to  $|G|$ .

By the result of [19], a minimal zero-sum sequence  $S = (x_1g) \cdot (x_2g) \cdot (x_3g) \cdot (x_4g)$  over  $G$  is reduced then  $|G|$  has at most two prime factors or one of the following holds:

- (A1)  $\{\gcd(x_i, n) | i = 1, 2, 3, 4\} = \{p_1, p_2, p_1p_3, p_2p_3\}$ ;
- (A2)  $\{\gcd(x_i, n) | i = 1, 2, 3, 4\} = \{1, p_1, p_2, p_1p_2\}$ ;
- (A3)  $\gcd(x_i, n) = 1$  for  $i = 1, 2, 3, 4$ ;
- (A4)  $\gcd(x_1, n) = 1, \gcd(x_2, n) = p_1p_2, \gcd(x_3, n) = p_1p_3, \gcd(x_4, n) = p_2p_3$ .

In this paper, we give the affirmative proof under assumption (A1), and our main result can be stated by the following theorem:

**Theorem 1.3.** *Let  $G = \langle g \rangle$  be a finite cyclic group such that  $|G| = p_1p_2p_3$  and  $\gcd(n, 6) = 1$ . If  $S = (x_1g, x_2g, x_3g, x_4g)$  is a minimal zero-sum sequence over  $G$  such that*

$$\{\gcd(n, x_i) | i = 1, 2, 3, 4\} = \{p_1, p_2, p_1p_3, p_2p_3\}.$$

*Then  $\text{ind}(S) = 1$ .*

It was mentioned in [13] that Conjecture 1.2 was confirmed computationally if  $n \leq 1000$ . Hence, throughout the paper, we always assume that  $n > 1000$ .

## 2. PRELIMINARIES AND RENUMBERING THE SEQUENCE

Throughout, let  $G$  be a cyclic group of order  $|G| = n > 1000$ . Given real numbers  $a, b \in \mathbb{R}$ , we use  $[a, b] = \{x \in \mathbb{Z} | a \leq x \leq b\}$  to denote the set of integers between  $a$  and  $b$ , and similarly, set  $[a, b) = \{x \in \mathbb{Z} | a \leq x < b\}$ . For  $x \in \mathbb{Z}$ , we denote by  $|x|_n \in [1, n]$  the integer congruent to  $x$  modulo  $n$ . Suppose that  $n$  has a prime decomposition  $n = p^\alpha q^\beta$ . Let  $S = (x_1g) \cdot \dots \cdot (x_4g)$  be a minimal zero-sum sequence over  $G$  such that  $\text{ord}(g) = n = |G|$  and  $1 \leq x_1, x_2, x_3, x_4 \leq n - 1$ . Then  $x_1 + x_2 + x_3 + x_4 = \nu n$ , where  $1 \leq \nu \leq 3$ .

Let  $S$  be the sequence as described in Theorem 1.3. Similar to Remark 2.1 of [11], we may always assume that  $x_1 = e, e + x_2 + x_3 + x_4 = 2n$  and  $e < x_2 < \frac{n}{2} < x_3 \leq x_4 < n - e$ . Let  $c = x_2, b = n - x_3, a = n - x_4$ , then it is easy to show that the following proposition implies Theorem 1.3.

**Proposition 2.1.** *Let  $n = p_1p_2p_3$ , where  $p_1, p_2, p_3$  are three different primes, and  $\gcd(n, 6) = 1$ . Let  $S = (g) \cdot (cg) \cdot ((n - b)g) \cdot ((n - a)g)$  be a minimal zero-sum sequence over  $G$  such that*

$\text{ord}(g) = |G| = n$ , and

$$\{\gcd(n, e), \gcd(n, c), \gcd(n, b), \gcd(n, a)\} = \{p_1, p_2, p_1 p_3, p_2 p_3\},$$

where  $e + c = a + b$ . Then  $\text{ind}(S) = 1$ .

*Notice that:* for convenience, we list two sufficient conditions introduced in Remark 2.1 of [11].

(1) If there exists positive integer  $m$  such that  $\gcd(n, m) = 1$  and  $|mx_1|_n + |mx_2|_n + |mx_3|_n + |mx_4|_n = 3n$ , then  $\text{ind}(S) = 1$ .

(2) If there exists positive integer  $m$  such that  $\gcd(n, m) = 1$  and at most one  $|mx_i|_n \in [1, \frac{n}{2}]$  (or, similarly, at most one  $|mx_i|_n \in [\frac{n}{2}, n]$ ), then  $\text{ind}(S) = 1$ .

**Lemma 2.2.** *Proposition 2.1 holds if one of the following conditions holds:*

(1) *There exist positive integers  $k, m$  such that  $\frac{kn}{c} \leq m \leq \frac{kn}{b}$ ,  $\gcd(m, n) = 1$ ,  $1 \leq k \leq b$ , and  $ma < n$ .*

(2) *There exists a positive integer  $M \in [1, \frac{n}{2e}]$  such that  $\gcd(M, n) = 1$  and at least two of the following inequalities hold:*

$$|Ma|_n > \frac{n}{2}, |Mb|_n > \frac{n}{2}, |Mc|_n < \frac{n}{2}.$$

**Lemma 2.3.** *If there exist integers  $k$  and  $m$  such that  $\frac{kn}{c} \leq m \leq \frac{kn}{b}$ ,  $\gcd(m, n) = 1$ ,  $1 \leq k \leq b$ , and  $a \leq \frac{b}{k}$ , then Proposition 3.1 holds.*

From now on, we assume that  $s = \lfloor \frac{b}{a} \rfloor$ . Then we have  $1 \leq s \leq \frac{b}{a} < s + 1$ . Since  $b \leq \frac{n}{2}$ , we have  $\frac{n}{2b} = \frac{(2s-t)n}{2b} - \frac{(2s-t-1)n}{2b} > 1$ , and then  $[\frac{(2s-t-1)n}{2b}, \frac{(2s-t)n}{2b}]$  contains at least one integer for every  $t \in [0, s-1]$ .

**Lemma 2.4.** *Suppose that  $a > 2e$ ,  $s \geq 2$  and  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains an integer co-prime to  $n$  for some  $t \in [0, \dots, \lfloor \frac{s}{2} \rfloor - 1]$ . Then Proposition 2.1 holds.*

For the proof of Lemma 3.3, Lemma 3.4 and Lemma 3.5, one is referred to the proof of Lemma 2.3-2.5 in [11], and we omit it here.

Let  $\Omega$  denote the set of those integers:  $x \in \Omega$  if and only if  $x \in [\frac{(2s-t-1)n}{2b}, \frac{(s-t)n}{b}]$  for some  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ . By Lemma 3.5, we also assume that

(B):  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains no integers co-prime to  $n$  for every  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ .

**Lemma 2.5.** *Suppose that  $a > 2e$ ,  $s \geq 2$  and  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains no integers co-prime to  $n$  for every  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ . Then  $[\frac{(2s-t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains at most 3 integers for every  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ . Hence  $\frac{n}{2b} < 4$ .*

**Lemma 2.6.** *Suppose that  $a > 2e$ ,  $s \geq 4$  and  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains no integers co-prime to  $n$  for every  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ . Then  $[\frac{(2s-2t-1)n}{2b}, \frac{(s-t)n}{b}]$  contains at most two integers for every  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$  and  $\frac{n}{2b} < 3$ .*

**Lemma 2.7.** *Suppose that  $a > 2e$  and  $s \geq 6$ , then there exists  $t_1 \in \{0, \lfloor \frac{s}{2} \rfloor - 1\}$  such that  $[\frac{(2s-t_1-1)n}{2b}, \frac{(2s-t_1)n}{2b}]$  contains exactly one integer and  $\frac{n}{2b} < 2$ .*

**Lemma 2.8.** *Suppose that  $a > 2e$  and  $s \geq 8$ , then  $[\frac{(2s-2t-1)n}{2b}, \frac{(2s-t)n}{b}]$  contains exactly one integer for every  $t \in [0, \lfloor \frac{s}{2} \rfloor - 1]$ .*

**Lemma 2.9.** *Under assumption  $a > 2e$ , we have  $s \leq 9$ .*

For the proof of Lemma 2.2-2.9 and more details, one is referred to [20], Li and Peng's paper [11] is also recommended.

Out of question, we can assume that  $e = \min\{p_1, p_2\}$ , without loss of generality, let  $p_1 < p_2$ , then  $e = p_1$ .

**Lemma 2.10.** *If  $a < 4e$ , then  $p_3 < p_1 = e < p_2$  and  $a = kp_2$  for some  $k \in [1, 3]$ .*

*Proof.* Since  $p_2 > p_1$  and  $p_2p_3 > p_1p_3 \geq 5p_1$ , it must hold that  $a = kp_2$  for some  $k \in [1, 3]$ . Hence we only need prove the case  $p_3 > p_1$ .

If  $p_3 > p_1$  and  $a = 3p_2$ , it holds that  $p_3|(c - b) = (a - e) = 3p_2 - p_1 \in 2\mathbb{Z}$ . If  $3p_2 - p_1 \geq 4p_3$ , we have  $a \geq 4p_3 + p_1 > 5p_1 = 5e$ , a contradiction. Then  $3p_2 - p_1 = 2p_3$  and  $p_3 > p_2$ . Simply computing shows that  $p_1 \geq 29$  and  $p_3 \geq 41$ . Then  $a = 3p_2 > 2p_1 = 2e$  and  $\frac{b}{a} \geq \frac{p_1p_3}{3p_2} > \frac{p_3}{4} > 10$ , which contradicts to the results of Lemma 2.9.

If  $p_3 > p_1$  and  $a = p_2$ , it holds that  $p_3|(c - b) = (a - e) = p_2 - p_1 \in 2\mathbb{Z}$ . If  $p_2 - p_1 \geq 4p_3$ , we have  $a > 5e$ , a contradiction. Then  $p_2 - p_1 = 2p_3$ . Applying Lemma 2.9, similar to above, we have  $b = 2p_1p_3$ ,  $p_1 = 11$ ,  $p_3 = 13$ ,  $p_2 = 37$ , or  $b = p_1p_3$ . If  $p_1 = 11$ ,  $p_3 = 13$ ,  $p_2 = 37$  and  $b = 2p_1p_3$ , then  $p_2p_3|c = b + a - e = 2p_1p_3 + p_2 - p_1 = 312 < 481 = p_2p_3$ , a contradiction. If  $b = p_1p_3$ , then  $p_2p_3|c = b + a - e = p_1p_3 + p_2 - p_1 \leq (p_2 - 4)p_3 + p_2 - p_1 < p_2p_3$ , which is a contradiction.

If  $p_3 > p_1$  and  $a = 2p_2$ , it holds that  $p_3|(c - b) = (a - e) = 2p_2 - p_1 \in 2\mathbb{Z} + 1$ . If  $2p_2 - p_1 \geq 3p_3$ , we have  $a > 4e$ , a contradiction. Then  $2p_2 - p_1 = p_3$ . Applying Lemma 2.9, similar to above, we have  $b = p_1p_3$  and  $p_1 = 7$ ,  $p_2 = 13$ ,  $p_3 = 19$  or  $p_1 = 11$ ,  $p_2 = 17$ ,  $p_3 = 23$ . Since  $p_2p_3|c = b + a - e = p_1(p_3 - 1) + p_2 < p_3p_2$ , we obtain a contradiction.  $\square$

**Lemma 2.11.** *If  $2e < a < 4e$ , then  $\text{ind}(S) = 1$ .*

*Proof.* By Lemma 2.10, it holds that  $a = kp_2$  for some  $k \in [1, 3]$ . We distinguish three cases according to the value of  $k$ .

**Case 1.**  $k = 1$ .

*Subcase 1.1.*  $p_2|c$ .

If  $c = p_2p_3$ , then  $p_1|c - a = p_2(p_3 - 1)$ , which implies  $p_1|(p_3 - 1) < p_1$ , a contradiction.

If  $c = 3p_2p_3$ , then  $\frac{b}{a} = \frac{c+e-a}{a} > 3p_3 - 1 \geq 14 > 10$ , a contradiction.

If  $c = 2p_2p_3$ , then  $p_1|c - a = p_2(2p_3 - 1)$ , which implies that  $p_1 = 2p_3 - 1$ . If  $p_3 \geq 7$ , we have  $\frac{b}{a} = \frac{c+e-a}{a} \geq 13 > 10$ , a contradiction. If  $p_3 = 5$ , then  $p_1|9$ , a contradiction.

*Subcase 1.2.*  $p_2|b$ .

If  $b = p_2p_3$ , then  $p_1|b + a = p_2(p_3 + 1)$ , which implies  $p_1 = p_3 + 1 < p_1$ , a contradiction.

If  $b \geq 2p_2p_3$ , then  $\frac{b}{a} \geq 10$ , a contradiction.

**Case 2.**  $k = 2$ .

*Subcase 2.1.*  $p_2|c$ .

If  $c = p_2p_3$ , then  $p_1|c - a = p_2(p_3 - 2)$ , which implies  $p_1|(p_3 - 2) < p_1$ , a contradiction.

If  $c = 2p_2p_3$ , then  $p_1|c - a = 2p_2(p_3 - 1)$ , which implies  $p_1|(p_3 - 1) < p_1$ , a contradiction.

If  $c = 3p_2p_3$ , then  $p_1|c - a = p_2(3p_3 - 2)$ . If  $p_3 \geq 11$ , we have  $\frac{b}{a} = \frac{c+e-a}{a} > 10$ , a contradiction. If  $p_3 = 5$ , then  $p_1 = 13$  and  $p_2 = 19$ ,  $b = c + e - a = 3 \times 19 \times 5 + 13 - 38 = 260$ . Since  $17 < \frac{4n}{c} = \frac{52}{3} < 18 < 19 = \frac{4n}{b}$  and  $\gcd(n, 18) = 1$ ,  $18a = 684 < n$ , let  $m = 18$  and  $k = 4$ , then  $\text{ind}(S) = 1$ . If  $p_3 = 7$ , then  $p_1 = 19$ . However, we can't find a prime  $p_2$  such that  $7|(2p_2 - 19)$  and  $19 < p_2 < 38$ .

*Subcase 2.2.  $p_2|b$ .*

If  $b = p_2p_3$ , then  $p_1|b + a = p_2(p_3 + 2)$ , which implies  $p_1 = p_3 + 2$ . By Lemma 2.9, we have  $p_3 \leq 19$ , in further,  $p_3 \in \{5, 11, 17\}$ . If  $p_3 = 5$ ,  $p_1 = 7$ ,  $p_2 \leq 13$ , which contradicts to  $n = p_1p_2p_3 > 1000$ . If  $p_3 = 11$ ,  $p_1 = 13$ , then  $p_2 \in \{17, 19\}$ , which contradicts to  $p_3|(2p_2 - p_1)$ . If  $p_3 = 17$ ,  $p_1 = 19$ , then  $p_2 \in \{23, 29, 31, 37\}$ , which also contradicts to  $p_3|(2p_2 - p_1)$ .

If  $b = 2p_2p_3$ , then  $p_1|b + a = 2p_2(p_3 + 1)$ , which implies  $p_1|p_3 + 1 < p_1$ , a contradiction.

If  $b = 3p_2p_3$ , then  $p_1|b + a = p_2(3p_3 + 2)$ . If  $p_3 \geq 7$ , we have  $\frac{b}{a} \geq \frac{23}{2} > 10$ , a contradiction. If  $p_3 = 5$ , then  $p_1 = 17$  and  $p_2 = 31$ ,  $c = b + a - e = 510$ . Since  $\frac{6n}{c} = 31 < 32 < 34 = \frac{6n}{b}$  and  $\gcd(n, 32) = 1$ ,  $32a = 1984 < 2635 = n$ , let  $m = 32$  and  $k = 6$ , then  $\text{ind}(S) = 1$ .

**Case 3.  $k = 3$ .**

*Subcase 3.1.  $p_2|c$ .*

If  $c = p_2p_3$ , then  $p_1|c - a = p_2(p_3 - 3)$ , which implies  $p_1|(p_3 - 2) < p_1$ , a contradiction.

If  $c = 3p_2p_3$ , then  $p_1|c - a = 3p_2(p_3 - 1)$ , which implies  $p_1|(p_3 - 1) < p_1$ , a contradiction.

If  $c = 2p_2p_3$ , then  $p_1|c - a = p_2(2p_3 - 3)$ , which implies  $p_1 = 2p_3 - 3$ . If  $p_3 \geq 17$ , we have  $\frac{b}{a} = \frac{c+e-a}{a} > 10$ , a contradiction. If  $p_3 = 5$ , then  $p_1 = 7$  and  $p_2 < 10$ , a contradiction. If  $p_3 = 7$ , then  $p_1 = 11$  and  $p_2 = 13$ , thus  $\frac{n}{c} < 6 < \frac{n}{b}$  and  $\gcd(n, 6) = 1$ . Let  $m = 6$  and  $k = 1$ , then  $\text{ind}(S) = 1$ . If  $p_3 = 11$ , then  $p_1 = 19$ , but there exists no prime  $p_2$  such that  $3p_2 < 4p_1$  and  $p_3|(3p_2 - p_1)$ . If  $p_3 = 13$ , then  $p_1 = 23$ , there exists no prime  $p_2$  such that  $3p_2 < 4p_1$  and  $p_3|(3p_2 - p_1)$ . If  $p_3 = 17$ , which implies  $p_1|49$  hence  $p_1 = 7 < p_3$ , a contradiction..

*Subcase 3.2.  $p_2|b$ .*

If  $b = p_2p_3$ , then  $p_1|b + a = p_2(p_3 + 3)$ , which implies  $2p_1|p_3 + 3$ , a contradiction. If  $b = 3p_2p_3$ , then  $p_1|b + a = 3p_2(p_3 + 1)$ , which implies  $p_1|p_3 + 1 < p_1$ , a contradiction.

If  $b = 2p_2p_3$ , then  $p_1|b + a = p_2(2p_3 + 3)$ . If  $p_3 \geq 17$ , we have  $\frac{b}{a} \geq \frac{37}{3} > 10$ , a contradiction. If  $p_3 = 5$ , then  $p_1 = 13$  and  $p_2 = 17$ , which contradicts to  $p_3|(3p_2 - p_1)$ . If  $p_3 = 7$ , then  $p_1 = 17$  and there exists no suitable  $p_2$ . If  $p_3 = 11$ , we can't find suitable  $p_1$ . If  $p_3 = 13$ , then  $p_1 = 29$ , we can't find suitable  $p_2$ .  $\square$

**Lemma 2.12.** *If  $a < 2e$  and  $a|b$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Let  $m = \frac{n+a}{a}$ ,  $m_1 = \frac{n+2a}{a}$ ,  $m_2 = \frac{n+3a}{a}$ ,  $m_3 = \frac{n+4a}{a}$ .

If  $\gcd(n, m) = 1$  then

$$|me|_n > \frac{n}{2}, |m(n-a)|_n = n-a > \frac{n}{2}, |m(n-b)|_n = n-b > \frac{n}{2},$$

thus  $\text{ind}(S) = 1$ .

Next assume that  $\gcd(n, m) > 1$ . Then  $\gcd(n, m_1) = \gcd(n, m_2) = \gcd(n, m_3) = 1$ . Moreover,

$$|m_1e|_n > \frac{n}{2}, |m_2e|_n > \frac{n}{2}, |m_3e|_n > \frac{n}{2}, |m_1a|_n < \frac{n}{2}, |m_2a|_n < \frac{n}{2}, |m_3a|_n < \frac{n}{2}.$$

If  $b < \frac{n}{4}$ , we have  $|m_1(n-b)|_n = n-2b > \frac{n}{2}$ . If  $\frac{n}{4} < b < \frac{n}{3}$ , we have  $|m_3(n-b)|_n = 2n-4b > \frac{n}{2}$ . If  $\frac{n}{3} < b < \frac{n}{2}$ , we have  $|m_2(n-b)|_n = 2n-3b > \frac{n}{2}$ . Then we can find an integer  $m_i$  such that  $\gcd(n, m_i) = 1$  and all of  $|m_i e|_n, |m_i(n-b)|_n, |m_i(n-a)|_n$  are larger than  $\frac{n}{2}$ , which implies that  $\text{ind}(S) = 1$ .  $\square$

*Renumbering the sequence:*

Now we begin to renumber the sequence such that  $e < \frac{a}{4}$ . For this purpose, by Lemma 2.10, 2.11 and Lemma 2.12, we can assume that  $a = p_2 < 2e$  and  $p_2 | c$ .

**Lemma 2.13.** *If  $a = p_2 < 2e$  and  $a | c$ , then  $\text{ind}(S) = 1$  or the sequence  $S$  can be renumbered as*

$$(e'g) \cdot (c'g) \cdot ((n-b')g) \cdot ((n-a')g)$$

*such that  $e' < a' \leq b' < c$  and  $a \geq 10e'$ . Moreover,  $e' = p_2$  or  $e' = 2p_2$ .*

*Proof.* Let  $m = \frac{n-a}{a}$ ,  $m_1 = \frac{n-2a}{a}$ ,  $m_2 = \frac{n+3a}{2a}$ ,  $m_3 = \frac{n+5a}{2a}$ .

If  $\gcd(n, m) = 1$ , then  $\frac{n}{2} < |me|_n < n-10a$  and  $|mc|_n = n-c > \frac{n}{2}$ . For this case, if  $|m(n-b)|_n > \frac{n}{2}$ , we have  $\text{ind}(S) = 1$ . Otherwise, it must hold  $a < |m(n-b)|_n$ . We get a renumbering:

$$(2.1) \quad e' = a, c' = |m(n-b)|_n, \{b', a'\} = \{c, n - |me|_n\},$$

and it is easy to check that  $a' \geq 10e'$ .

Next we assume that  $\gcd(m, n) > 1$ , then  $p_2 | (p_1 p_3 - 1)$  and  $\gcd(n, m_1) = \gcd(n, m_2) = \gcd(n, m_3) = 1$ .

If  $c = 2ta$  for some integer  $t$ . Let  $m' = \frac{n+a}{2a}$ . Then  $\gcd(n, m') = 1$  and  $|m'e|_n < \frac{n}{2}$ ,  $|m'c|_n = \frac{c}{2} < \frac{n}{2}$ ,  $|m'(n-a)|_n = \frac{n-a}{2} < \frac{n}{2}$ , and  $\text{ind}(S) = 1$ .

If  $c = (2t+1)a$  for some integer  $t$ . We distinguish three cases according to the value of  $\frac{n}{c}$ .

**Case 1.**  $\frac{n}{4} > c$ . Replace  $m$  by  $m_1$  and repeat the above process, we have  $|m_1(n-b)|_n > \frac{n}{2}$ ,  $|m_1c|_n > \frac{n}{2}$  and  $|m_1e|_n > \frac{n}{2}$ , which implies  $\text{ind}(S) = 1$ , or we can obtain a renumbering:

$$(2.2) \quad e' = 2a, c' = |m_1(n-b)|_n, \{b', a'\} = \{2c, n - |m_1e|_n\},$$

it also holds that  $a' \geq 10e'$ .

**Case 2.**  $\frac{n}{4} < c < \frac{n}{3}$ . Then  $|m_3(n-a)|_n = \frac{n-5a}{2} < \frac{n}{2}$ . We have  $|m_3e|_n < \frac{n}{2}$  and  $|m_3c|_n = |\frac{n+5c}{2}|_n < \frac{n}{2}$ , exactly it belongs to  $(\frac{n}{8}, \frac{n}{3})$ . Then  $\text{ind}(S) = 1$ .

**Case 3.**  $\frac{n}{3} < c$ . Then  $|m_2(n-a)|_n = \frac{n-3a}{2} < \frac{n}{2}$ . We have  $|m_2c|_n = |\frac{n+3c}{2}|_n < \frac{n}{4}$ ,  $|m_2e|_n < \frac{n}{2}$ , and hence  $\text{ind}(S) = 1$ .  $\square$

Through the process of renumbering, we can always assume that  $e \in \{p_1, p_2, 2p_2\}$  and  $a > 4e$ . Particularly,  $a \geq 10e$  when  $e \in \{p_2, 2p_2\}$ . Hence we also assume that  $s \leq 9$  by Lemma 2.9.

Let  $k_1$  be the largest positive integer such that  $\lceil \frac{(k_1-1)n}{c} \rceil = \lceil \frac{(k_1-1)n}{b} \rceil$  and  $\frac{k_1 n}{c} \leq m < \frac{k_1 n}{b}$  for some integer  $m$ . Since  $\frac{bn}{c} \leq n-1 < n = \frac{bn}{b}$  and  $\frac{tn}{b} - \frac{tn}{c} = \frac{t(c-b)n}{bc} > 2$  for all  $t \geq b$ , such integer  $k_1$  always exists and  $k_1 \leq b$ .

As mentioned above, we only need prove Proposition 2.1. We now show that Proposition 2.1 holds through the following 3 propositions.

**Proposition 2.14.** *Suppose  $\lceil \frac{n}{c} \rceil < \lceil \frac{n}{b} \rceil$ , then Proposition 2.1 holds.*

**Proposition 2.15.** Suppose  $\lceil \frac{n}{c} \rceil = \lceil \frac{n}{b} \rceil$ . Let  $k_1$  be the largest positive integer such that  $\lceil \frac{(k_1-1)n}{c} \rceil = \lceil \frac{(k_1-1)n}{b} \rceil$  and  $\frac{k_1 n}{c} \leq m_1 < \frac{k_1 n}{b}$  holds for some integer  $m_1$ . If  $k_1 > \frac{b}{a}$ , then Proposition 2.1 holds.

**Proposition 2.16.** Suppose  $\lceil \frac{n}{c} \rceil = \lceil \frac{n}{b} \rceil$ . Let  $k_1$  be the largest positive integer such that  $\lceil \frac{(k_1-1)n}{c} \rceil = \lceil \frac{(k_1-1)n}{b} \rceil$  and  $\frac{k_1 n}{c} \leq m_1 < \frac{k_1 n}{b}$  holds for some integer  $m_1$ . If  $k_1 \leq \frac{b}{a}$ , then Proposition 2.1 holds.

### 3. PROOF OF PROPOSITION 2.14

In this section, we assume that  $\lceil \frac{n}{c} \rceil < \lceil \frac{n}{b} \rceil$ . Let  $m_1 = \lceil \frac{n}{c} \rceil$ . Then we have  $m_1 - 1 < \frac{n}{c} \leq m_1 < \frac{n}{b}$ . By Lemma 2.3 (1), it suffices to find  $m$  and  $k$  such that  $\frac{kn}{c} \leq m < \frac{kn}{b}$ ,  $\gcd(m, n) = 1$ ,  $1 \leq k \leq b$ , and  $ma < n$ . So in what follows, we may always assume that  $\gcd(n, m_1) > 1$ .

**Lemma 3.1.** If  $\lceil \frac{n}{c}, \frac{n}{b} \rceil$  contains at least two integers, then  $\text{ind}(S) = 1$ .

*Proof.* Note that  $b \geq p_1 p_3$  and  $b \geq p_1 p_3$ . Thus  $\frac{n}{b} \leq p_2$  and  $\frac{n}{c} \leq p_1$ . It must hold

$$(3.1) \quad m_1 - 1 < \frac{n}{c} \leq m_1 < m_1 + 1 < \frac{n}{b} = m_1 + 2 = p_2,$$

or

$$(3.2) \quad m_1 - 1 < \frac{n}{c} \leq m_1 < m_1 + 1 \leq \frac{n}{b} < m_1 + 2 \leq p_2.$$

Clearly,  $p_1 = m_1$  or  $p_1 = m_1 + 1$ .

If (3.1) holds, we have  $m_1 = p_1, 2p_3 | (m_1 + 1), m_1 + 2 = p_2$  and  $b = p_1 p_3$ . Then we infer that  $c = p_2 p_3$  and  $a - e = 2p_3 < 2p_1 \leq 2e$ , which contradicts to the assumption  $a > 4e$ .

If (3.2) holds, we have  $m_1 \geq 10$  and

$$(3.3) \quad 2m_1 - 2 < \frac{2n}{c} \leq 2m_1 < 2m_1 + 1 < 2m_1 + 2 \leq \frac{n}{b} < 2m_1 + 4.$$

If  $\gcd(n, 2m_1 + 1) = 1$ , let  $m = 2m_1 + 1$  and  $k = 2$ , we have

$$ma < \frac{4m}{3}(a - e) = \frac{4m}{3}(c - b) < \frac{8m_1 + 4}{3} \times \frac{3n}{(m_1 + 2)(m_1 - 1)} \leq \frac{7n}{9} < n,$$

as desired. If  $\gcd(n, 2m_1 + 1) > 1$ , we infer that  $p_2 = 2m_1 + 1$  and  $m_1 \geq 28$ . Let  $m = 3m_1 + 2$  and  $k = 3$ , we have  $\gcd(n, m) = 1$  and

$$ma < \frac{4m}{3}(a - e) = \frac{4m}{3}(c - b) < \frac{12m_1 + 8}{3} \times \frac{3n}{(m_1 + 2)(m_1 - 1)} \leq \frac{172n}{405} < n,$$

and  $\text{ind}(S) = 1$ . □

By Lemma 3.1, we may assume that  $\lceil \frac{n}{c}, \frac{n}{b} \rceil$  contains exactly one integer  $m_1$ , and thus

$$(3.4) \quad m_1 - 1 < \frac{n}{c} \leq m_1 < \frac{n}{b} < m_1 + 1.$$

Let  $l$  be the smallest integer such that  $\lceil \frac{ln}{c}, \frac{ln}{b} \rceil$  contains at least four integers. Clearly,  $l \geq 3$ . Since  $\frac{n}{b} - m_1 < 1$  and  $m_1 - \frac{n}{c} < 1$ , by using the minimality of  $l$  we obtain that  $lm_1 - 4 < \frac{ln}{c} < \frac{ln}{b} < lm_1 + 4$ . Then  $\frac{ln(c-b)}{bc} = \frac{ln}{b} - \frac{ln}{c} < (lm_1 + 4) - (lm_1 - 4) = 8$  and thus

$$l < \frac{8bc}{(c-b)n} < \frac{8b}{(a-e)(m_1-1)} < \frac{8b}{3e(m_1-1)} < b.$$

We claim that  $[\frac{ln}{c}, \frac{ln}{b})$  contains at most six integers. For any positive integer  $j$ , let  $N_j$  denote the number of integers contained in  $[\frac{ln}{c}, \frac{ln}{b})$ . Since

$$\begin{aligned} \left( \frac{(j+1)n}{b} - (j+1)m_1 \right) - \left( \frac{jn}{b} - jm_1 \right) &= \frac{n}{b} - m_1 < 1, \\ \left( (j+1)m_1 - \frac{(j+1)n}{c} \right) - \left( jm_1 - \frac{jn}{c} \right) &= m_1 - \frac{n}{c} < 1, \end{aligned}$$

we infer that  $N_{j+1} - N_j \leq 2$ , it is sufficient to show our claim.

By the claim above we have

$$lm_1 - j_0 < \frac{ln}{c} \leq lm_1 - j_0 + 1 < \cdots < lm_1 - j_0 + 4 < \frac{ln}{b} \leq lm_1 - j_0 + 6$$

for some  $1 \leq j_0 \leq 4$ . We remark that since  $n = p_1 p_2 p_3$  and  $[\frac{ln}{c}, \frac{ln}{b})$  contains at least four integers, one of them (say  $m$ ) must be co-prime to  $n$ . If  $ma < n$ , then we have done by Lemma 2.2(1) (with  $k = l < b$ ).

Proposition 2.14 can be proved by the following three lemmas.

**Lemma 3.2.** *If  $m_1 \neq 5, 7$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Since  $m_1 \neq 5, 7$ , we have  $m_1 \geq 10$  and  $n > m_1 b \geq 10b$ . Let  $k = l$  and let  $m$  be one of the integers in  $[\frac{ln}{c}, \frac{ln}{b})$  which is co-prime to  $n$ . Note that  $m \leq lm_1 + 3$  and  $l \geq 3$ , then

$$ma \leq (lm_1 + 3)a < \frac{4(lm_1 + 3)}{3} \left( \frac{ln}{lm - j_0} - \frac{ln}{lm - j_0 + 6} \right) = \frac{4(lm_1 + 3) \times 6ln}{3(lm_1 - j_0)(lm_1 - j_0 + 6)} < n.$$

and we have done.  $\square$

**Lemma 3.3.** *If  $m_1 = 5$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Since  $m_1 = 5$ , we have  $4 < \frac{n}{c} \leq 5 < \frac{n}{b} < 6$ , thus  $a < \frac{4}{3}(c - b) < \frac{n}{9}$ . If  $\frac{2n}{c} < 9 < \frac{2n}{b}$ , let  $m = 9$  and  $k = 2$ , then we have done. Then  $9 < \frac{2n}{c}$ , so  $a < \frac{4}{3}(c - b) < \frac{2n}{27} < \frac{n}{13}$ .

If  $\frac{2n}{c} < 11 < \frac{2n}{b}$ , we infer that  $\frac{27}{2} < \frac{3n}{c} \leq 15 < 16 < \frac{33}{2} < \frac{3n}{b} < 18$ . If  $16a < n$ , let  $m = 16$  and  $k = 3$ , we have done. If  $16a > a$ , then  $n < 18a < 2n$ ,  $3n < 18b < 18c < 4n$  and  $18e < \frac{9a}{2} < n$ , let  $M = 18$ . Then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 3n) + (4n - Mb) + (2n - Ma) = 3n$ , then we have done.

Next we assume that  $9 < \frac{2n}{c} \leq 10 < \frac{2n}{b} \leq 11$ .

We infer that  $a < \frac{n}{18}$ . If  $\frac{3n}{c} < 16 < \frac{3n}{b}$ , let  $m = 16$  and  $k = 3$ , then we have done. Otherwise,  $\frac{27}{2} < \frac{3n}{c} \leq 15 < \frac{3n}{b} < 16$ , hence we obtain  $a < \frac{n}{21}$ .

If  $\frac{27}{2} < \frac{3n}{c} \leq 14 < 15 < \frac{3n}{b} < 16$ , we have  $\frac{3n}{16} < b < \frac{n}{5} < \frac{3n}{14} < c < \frac{2n}{9}$ . If  $24a < n$ , let  $M = 12$ , we have  $|M(n-a)|_n > \frac{n}{2}$ ,  $|M(n-b)|_n > \frac{n}{2}$ ,  $|Mc|_n > \frac{n}{2}$ , then  $\text{ind}(S) = 1$ . If  $24a > n$ , we infer that  $n < 27a < 2n$ ,  $27e < n$  and  $5n < \frac{81n}{16} < 27b < 27c < 6n$ , let  $M = 27$ . Then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 5n) + (6n - Mb) + (2n - Ma) = 3n$ , and we have done.

Next assume that  $14 < \frac{3n}{c} \leq 15 < \frac{3n}{b} < 16$ , we infer that  $28a < n$ .

If  $\frac{4n}{c} \leq 19 < 20 < 21 < \frac{4n}{b}$ , since either 19 or 21 is co-prime to  $n$  (otherwise,  $n = 5 \times 7 \times 19 = 665 < 1000$ , a contradiction), let  $m$  be one of 19, 21 such that  $\gcd(n, m) = 1$  and  $k = 4$ , then we have done.



If  $\frac{4n}{c} \leq 19 < 20 < \frac{4n}{b} \leq 21$ , let  $M = 24$ , then  $4n < 24b < \frac{24n}{5} < 5n < \frac{96n}{19} < 24 \times \frac{4n}{19} < 24c < 6n$ , and  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 5n) + (5n - Mb) + (n - Ma) = n$ , and we have done.

If  $19 < \frac{4n}{c} \leq 20 < 21 < \frac{4n}{b}$ , if  $\gcd(n, 21) = 1$ , let  $m = 21$  and  $k = 4$ , then we have done. If  $\gcd(n, 21) > 1$ , then  $\gcd(26, n) = 1$ . Otherwise,  $n = 5 \times 7 \times 13 < 1000$ , a contradiction. Let  $M = 26$ , we have  $4n < 26b < \frac{104n}{21} < 5n < \frac{26n}{5} < 26c < 6n$ , and  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 5n) + (5n - Mb) + (n - Ma) = n$ . Then  $\text{ind}(S) = 1$ .

Next assume that  $19 < \frac{4n}{c} \leq 20 < \frac{4n}{b} \leq 21$ .

If  $39 < \frac{8n}{c} \leq 40 < \frac{8n}{b} \leq 41$ , we infer that  $a < \frac{n}{73}$ . Then  $b > \frac{n}{6} > \frac{73a}{6} > 12a$ , which contradicts to  $s \leq 9$ .

If  $39 < \frac{8n}{c} \leq 40 < 41 < \frac{8n}{b} \leq 42$ , we infer that  $a < \frac{n}{51}$ . Let  $M = 36$ , then  $|Me|_n < 9a < \frac{n}{2}$ ,  $|Mc|_n < \frac{n}{2}$  and  $|M(n-a)|_n < \frac{n}{2}$  (otherwise,  $72a < n$  contradicts to  $s \leq 9$ ). Exactly,  $|Mc|_n$  belongs to  $(\frac{n}{5}, \frac{15n}{39})$ . Then  $\text{ind}(S) = 1$ .

If  $38 < \frac{8n}{c} \leq 39 < 40 < \frac{8n}{b} \leq 41$ , we infer that  $\frac{n}{52} < a < \frac{n}{48}$ . If  $\gcd(n, 39) = 1$ , let  $m = 39$  and  $k = 8$ , then we have done. If  $\gcd(n, 39) > 1$ , then  $\gcd(n, 11) = 1$ , otherwise  $n = 5 \times 11 \times 13 = 715 < 1000$ , a contradiction. Let  $M = 44$ , then  $\gcd(n, M) = 1$  and  $|M(n-a)|_n < \frac{n}{2}$  (otherwise,  $88a < n$  contradicts to  $s \leq 9$ ),  $|M(n-b)|_n < \frac{n}{2}$ ,  $|Mc|_n < \frac{n}{2}$ . Exactly,  $|M(n-b)|_n$  belongs to  $(\frac{n}{5}, \frac{13n}{41})$  and  $|Mc|_n$  belongs to  $(\frac{n}{39}, \frac{3n}{19})$ . Then  $\text{ind}(S) = 1$ .

Now we infer that  $38 < \frac{8n}{c} \leq 39 < 40 < 41 < \frac{8n}{b} \leq 42$  and  $\frac{n}{53} < a < \frac{n}{37}$ . Let  $M = 36$ , we have  $|M(n-a)|_n < \frac{n}{2}$  (otherwise,  $72a < n$  contradicts to  $s \leq 9$ ,  $Me < 9a < \frac{n}{2}$  and  $6n + \frac{6n}{7} \leq Mb < 7n + \frac{n}{41}$ . If  $Mb < 7n$ , then  $|M(n-b)|_n < \frac{n}{2}$  and  $\text{ind}(S) = 1$ . Hence we infer that  $a < \frac{n}{46}$ . Moreover, we can assume that  $n = 5 \times 13 \times 41$ . Otherwise, there exists an integer (say  $m$ ) between 39 and 41 such that  $\gcd(n, m) = 1$  and  $ma < n$ . Let  $k = 8$ , thus  $\text{ind}(S) = 1$ . Simply calculating shows that  $p_1 = 5, p_2 = 13$ . However, we can't find suitable  $a$  and  $e$  such that  $a > 4e$ .

Hence we complete the proof.  $\square$

**Lemma 3.4.** *If  $m_1 = 7$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Since  $m_1 = 7$ , we have  $6 < \frac{n}{c} \leq 7 < \frac{n}{b} < 8$ , thus  $a < \frac{4}{3}(c-b) < \frac{n}{18}$ .

If  $12 < \frac{2n}{c} \leq 13 < 14 < 15 < \frac{n}{b} < 16$ , then at least one of 13, 14, 15 is co-prime to  $n$ . Let  $m \in [13, 15]$  such that  $\gcd(n, m) = 1$  and  $k = 1$ , then  $ma < n$  and  $\text{ind}(S) = 1$ .

If  $13 < \frac{2n}{c} \leq 14 < 15 < \frac{2n}{b} < 16$ , we have  $\frac{n}{8} < b < \frac{2n}{15} < \frac{n}{7} < c < \frac{2n}{13}$ ,  $a < \frac{n}{26}$  and  $2n < 16b < 2n + \frac{2n}{15} < 2n + \frac{2n}{7} < 16c < 2n + \frac{6n}{13}$ . If  $32a > n$ , let  $M = 16$ , we have  $|Me|_n < \frac{n}{2}$ ,  $|Mc|_n < \frac{n}{2}$  and  $|M(n-a)|_n < \frac{n}{2}$ , then  $\text{ind}(S) = 1$ . If  $32a < n$ , by inequality  $\frac{4n}{c} \leq 28 < 29 < 30 < \frac{4n}{b}$ , we infer that  $n = 5 \times 7 \times 29$ . Since  $e < \frac{a}{4} < \frac{n}{104}$ , we have  $e < 10$ , then  $e = p_1$ . If  $p_1 = 7$ , then  $p_2 = 29$  and  $c = 5 \times 29$ ,  $a = 29$ , thus  $b = 4 \times 29 + 7$ , which contradicts to  $7|b$ . We infer that  $p_1 = 5$ , and thus  $\frac{n}{c} \leq \frac{n}{p_2 p_3} = 5$ , a contradiction.

If  $12 < \frac{2n}{c} \leq 13 < 14 < \frac{2n}{b} < 15$ , we infer that  $a < \frac{n}{22}$  and  $91|n$ . we also assume that  $27a > n$ . Otherwise, let  $m = 27$  and  $k = 4$ , we have  $\frac{4n}{c} \leq 26 < 27 < 28 < \frac{42n}{b}$ . If  $5|n$ , then  $n = 5 \times 91 = 455 < 1000$ , a contradiction. Thus  $\gcd(n, 30) = 1$ . Let  $M = 30$ , we have  $Me < 8a < n$ ,  $4n < 30b < 30c < 5n$  and  $n < 30a < 2n$ . Then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 4n) + (5n - Mb) + (2n - Ma) = 3n$ , and  $\text{ind}(S) = 1$ .

Next assume that  $13 < \frac{2n}{c} \leq 14 < \frac{2n}{b} < 15$ , and we infer that  $a < \frac{n}{36}$ .

If  $\frac{4n}{c} < 27 < \frac{4n}{b}$ , let  $m = 27$  and  $k = 4$ , then we have done. So  $27 < \frac{4n}{c} \leq 28 < \frac{4n}{b} < 30$ , and  $a < \frac{n}{50}$ .

If  $27 < \frac{4n}{c} \leq 28 < 29 < \frac{4n}{b} < 30$ , we have  $\frac{2n}{15} < b < \frac{4n}{29} < \frac{n}{7} \leq c < \frac{4n}{27}$  and  $\frac{24n}{5} < b < \frac{144n}{29} < 5n < \frac{36n}{7} \leq c < \frac{48n}{9} < 6n$ . Let  $M = 36$ , we have  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 5n) + (5n - Mb) + (n - Ma) = n$ , and  $\text{ind}(S) = 1$ .

If  $27 < \frac{4n}{c} \leq 28 < \frac{4n}{b} \leq 29$ , we infer that  $a < \frac{n}{73}$ , then  $b > \frac{4n}{29} > \frac{292a}{29} > 10a$ , which contradicts to  $s \leq 9$ .

We complete the proof.  $\square$

#### 4. PROOF OF PROPOSITION 2.15

In this section, we always assume that  $\lceil \frac{n}{c} \rceil = \lceil \frac{n}{b} \rceil$ , so  $k_1 \geq 2$ , and we also assume that  $k_1 > \frac{b}{a}$ . Proposition 2.15 can be proved through the following two lemmas.

**Lemma 4.1.** *If the assumption is as in Proposition 2.15, then  $k_1 < 3$ .*

*Proof.* If  $k_1 \geq 3$ , then  $\frac{(k_1-1)n}{b} - \frac{(k_1-1)n}{c} = \frac{(a-e)(k_1-1)n}{bc} \geq \frac{3a}{4} \frac{2k_1n}{3bc} > 1$ , a contradiction.  $\square$

**Lemma 4.2.** *If the assumption is as in Proposition 2.15 and  $k_1 = 2$ , then  $\text{ind}(S) = 1$ .*

*Proof.* If  $\frac{n}{c} > 3$ , then  $\frac{n}{b} - \frac{n}{c} = \frac{(a-e)n}{bc} \geq \frac{2a}{3} \frac{n}{bc} > 1$ , a contradiction.

If  $\frac{n}{c} \leq 3 < \frac{n}{b}$ , we have  $n < 3c < 2n$ ,  $3a < 3b < n$ . Let  $m = 3$ , then  $\gcd(n, m) = 1$  and  $|me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n = me + (mc - n) + (n - mb) + (n - ma) = n$ , we have done.

If  $\frac{n}{c} < \frac{n}{b} < 3$ , then  $\frac{n}{3} < b < 2a$ , and  $2n < 6c < 3n$ ,  $2n < 6b < 3n$ ,  $6a > 3b > n$ .  $6e < 2a < n$ . Let  $m = 6$ , then  $\gcd(n, m) = 1$ , and  $3n \geq |me|_n + |mc|_n + |m(n-b)|_n + |m(n-a)|_n \geq me + (mc - 2n) + (3n - mb) + (2n - ma) = 3n$ , we have done.  $\square$

#### 5. PROOF OF PROPOSITION 2.16

In this section, we always assume that  $\lceil \frac{n}{c} \rceil = \lceil \frac{n}{b} \rceil$ , so  $k_1 \geq 2$ , and we also assume that  $k_1 < \frac{b}{a}$ , hence  $s \geq k_1$ . Proposition 2.16 can be proved by the following Lemmas 5.1-5.6 and Lemma 5.9.

**Lemma 5.1.** *If the assumption is as in Proposition 2.16, then  $k_1 \leq 7$ .*

*Proof.* If  $k_1 \geq 8$ , then  $\frac{(k_1-1)n}{b} - \frac{(k_1-1)n}{c} \geq \frac{(a-e)7n}{bc} \geq \frac{21an}{4bc} > \frac{21}{20} > 1$ , a contradiction.  $\square$

**Lemma 5.2.** *If the assumption is as in Proposition 2.16 and  $k_1 = 7$ , then  $\text{ind}(S) = 1$ .*

*Proof.* If  $c \leq \frac{9n}{20}$ , we have  $\frac{(k_1-1)n}{b} - \frac{(k_1-1)n}{c} \geq \frac{(a-e)6n}{bc} \geq \frac{9an}{2bc} \geq \frac{10a}{b} > 1$ , a contradiction. Thus  $\frac{n}{c} < \frac{20}{9}$  and we infer that  $12 < \frac{6n}{c} < \frac{6n}{b} \leq 13$  or  $13 < \frac{6n}{c} < \frac{6n}{b} \leq 14$ .

**Case 1.** It holds that  $12 < \frac{6n}{c} < \frac{6n}{b} \leq 13$ . Then  $a < \frac{n}{19}$  and we have

$$\begin{aligned} 14 &< \frac{7n}{c} \leq 15 < \frac{7n}{b} \leq \frac{91}{6}, \\ 16 &< \frac{8n}{c} \leq 17 < \frac{8n}{b} \leq \frac{52}{3}, \\ 18 &< \frac{9n}{c} \leq 19 < \frac{9n}{b} \leq \frac{39}{2}, \\ 20 &< \frac{10n}{c} \leq 21 < \frac{10n}{b} \leq \frac{65}{3}. \end{aligned}$$

If  $\gcd(n, 15) = 1$ , let  $m = 15$  and  $k = 7$ , then we have done. If  $\gcd(n, 17) = 1$ , let  $m = 17$  and  $k = 8$ , then we have done. If  $\gcd(n, 19) = 1$ , let  $m = 19$  and  $k = 9$ , then we have done.

Now assume that  $n = 5 \times 17 \times 19$  and thus  $\gcd(n, 21) = 1$ . If  $21a < n$ , let  $m = 21$  and  $k = 10$ , then we have done. If  $21a > n$ , let  $M = 12$ , we have  $|M(n-a)|_n < \frac{n}{2}$  and  $|Me|_n < \frac{n}{2}$ . Moreover,  $5n + \frac{7n}{13} = \frac{72n}{13} < 12b < \frac{28n}{5} = 5n + \frac{3n}{5}$ , which implies that  $|M(n-b)|_n < \frac{n}{2}$ . So  $\text{ind}(S) = 1$ .

**Case 2.** It holds that  $13 < \frac{6n}{c} < \frac{6n}{b} \leq 14$ . Then  $a < \frac{n}{22}$  and  $\frac{91}{6} < \frac{7n}{c} < 16 < \frac{7n}{b} \leq \frac{49}{3}$ . Let  $m = 16$  and  $k = 7$ , we have  $\gcd(n, m) = 1$  and  $ma < n$ , then  $\text{ind}(S) = 1$ .  $\square$

**Lemma 5.3.** *If the assumption is as in Proposition 2.16 and  $k_1 = 6$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Similar to Lemma 5.2, we have  $\frac{n}{c} < \frac{8}{3}$  and we infer that  $10 < \frac{5n}{c} < \frac{5n}{b} \leq 11$ , or  $11 < \frac{5n}{c} < \frac{5n}{b} \leq 12$ , or  $12 < \frac{5n}{c} < \frac{5n}{b} \leq 13$ , or  $13 < \frac{5n}{c} < \frac{5n}{b} \leq \frac{40}{3}$ .

**Case 1.** It holds that  $10 < \frac{5n}{c} < \frac{5n}{b} \leq 11$ , then  $a < \frac{n}{16}$ . If  $18a > n$ . Let  $M = 9$ , we infer that  $|M(n-a)|_n < \frac{n}{2}$ ,  $|Me|_n < \frac{n}{2}$  and  $|Mc|_n < \frac{n}{2}$  (exactly,  $4n + \frac{n}{11} < 9c < 4n + \frac{n}{2}$ ). Then  $\text{ind}(S) = 1$ . Moreover, we have

$$\frac{6n}{c} \leq 13 < \frac{6n}{b}, \quad \frac{7n}{c} \leq 15 < \frac{7n}{b}, \quad \frac{8n}{c} \leq 17 < \frac{8n}{b}.$$

If  $\gcd(n, 13) = 1$ , let  $m = 13$  and  $k = 6$ , if  $\gcd(n, 15) = 1$ , let  $m = 15$  and  $k = 7$ , if  $\gcd(n, 17) = 1$ , let  $m = 17$  and  $k = 8$ , then  $ma < n$  and  $\text{ind}(S) = 1$ . If none of the three integers is co-prime to  $n$ , then  $n = 5 \times 13 \times 17$  and  $p_1 = 13, p_2 = 17, p_3 = 5$ .

By the renumbering process, we may assume that  $17 \leq e \leq \frac{a}{10}$  or  $a \geq 4 \times 17$ . If  $17 \leq e \leq \frac{a}{10}$ , then  $e \leq \frac{n}{180} < 10$ , a contradiction. If  $a \geq 4 \times 17$ , then  $a \geq \frac{4n}{5 \times 13} > \frac{n}{17}$ , a contradiction.

**Case 2.** It holds that  $11 < \frac{5n}{c} < \frac{5n}{b} \leq 12$ . Then  $a < \frac{4}{3}(a-e) = \frac{4}{3}(c-b) < \frac{4}{3}(\frac{5n}{11} - \frac{n}{12}) < \frac{n}{19}$ . If  $\frac{8n}{c} < 18 < \frac{8n}{b}$ , let  $m = 18$  and  $k = 8$ , then we have done. Otherwise, it holds  $18 < \frac{8n}{c} \leq 19 < \frac{8n}{b} \leq \frac{96}{5}$ , and thus  $a < \frac{4}{3}(\frac{4n}{9} - \frac{5}{12}) = \frac{n}{27}$ . We infer that  $\frac{b}{a} > \frac{5n}{12} \times \frac{27}{n} > 10$ , which contradicts to  $s \leq 9$ .

**Case 3.** It holds that  $12 < \frac{5n}{c} < \frac{5n}{b} \leq 13$ . Then  $a < \frac{4}{3}(a-e) = \frac{4}{3}(c-b) < \frac{4}{3}(\frac{5n}{12} - \frac{n}{13}) < \frac{n}{23}$ . If  $\frac{7n}{c} < 18 < \frac{7n}{b}$ , let  $m = 18$  and  $k = 7$ , then we have done. Otherwise, it holds  $\frac{84}{5} < \frac{7n}{c} \leq 17 < \frac{7n}{b} \leq 18$ , and thus  $a < \frac{4}{3}(\frac{5n}{12} - \frac{7}{18}) = \frac{n}{27}$ . We infer that  $\frac{b}{a} > \frac{7n}{18} \times \frac{27}{n} > 10$ , which contradicts to  $s \leq 9$ .

**Case 4.** It holds that  $13 < \frac{5n}{c} < \frac{5n}{b} \leq \frac{40}{3}$ . Then  $a < \frac{4}{3}(a-e) = \frac{4}{3}(c-b) < \frac{4}{3}(\frac{5n}{13} - \frac{3n}{8}) = \frac{n}{78}$ . Since  $s \geq k_1 = 6$ , by Lemma 2.7, we have  $b > \frac{n}{4}$ , thus  $\frac{b}{a} > \frac{78}{4} > 19$ , which contradicts to  $s \leq 9$ .  $\square$

**Lemma 5.4.** *If the assumption is as in Proposition 2.16 and  $k_1 = 5$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Similar to Lemma 5.2, we have  $\frac{n}{c} < \frac{10}{3}$ , then  $8 + t < \frac{4n}{c} < \frac{4n}{b} \leq 9 + t$  for some  $t \in [0, 4]$  or  $13 < \frac{4n}{c} < \frac{4n}{b} \leq \frac{40}{3}$ . We distinguish six cases.

**Case 1.** It holds that  $13 < \frac{4n}{c} < \frac{4n}{b} \leq \frac{40}{3}$ . We have  $39 < \frac{12n}{c} < \frac{12n}{b} \leq 40$ , which contradicts to the maximality of  $k_1$ .

**Case 2.**  $t = 0$ .

It holds that  $8 < \frac{4n}{c} < \frac{4n}{b} \leq 9$ , and we infer that  $a < \frac{n}{13}$ . Moreover, we have

$$\frac{5n}{c} \leq 11 < \frac{5n}{b}, \quad \frac{6n}{c} \leq 13 < \frac{6n}{b}, \quad \frac{7n}{c} \leq 15 < \frac{7n}{b} < 16.$$

If  $16a > n$ , let  $M = 16$ , then  $Me < n$ ,  $n < Ma < 2n$  and  $7n < Mb < Mc < 8n$ . We infer that  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = 3n$ , and  $\text{ind}(S) = 1$ . Thus  $16a < n$ . If  $\gcd(n, 11) = 1$ , let  $m = 11$  and  $k = 5$ , if  $\gcd(n, 13) = 1$ , let  $m = 13$  and  $k = 6$ , if  $\gcd(n, 15) = 1$ , let  $m = 15$  and  $k = 7$ , then  $ma < n$  and  $\text{ind}(S) = 1$ . If none of 11, 13, 15 is co-prime to  $n$ , then  $n = 5 \times 11 \times 13 = 715 < 1000$ , a contradiction.

**Case 3.**  $t = 1$ .

It holds that  $9 < \frac{4n}{c} < \frac{4n}{b} \leq 10$ . We infer that  $a < \frac{n}{16}$  and  $\frac{45}{4} < \frac{5n}{c} < 12 < \frac{5n}{b} \leq \frac{25}{2}$ . Let  $m = 12$  and  $k = 5$ , then we have done.

**Case 4.**  $t = 2$ .

It holds that  $10 < \frac{4n}{c} < \frac{4n}{b} \leq 11$ . We infer that  $a < \frac{n}{20}$  and  $15 < \frac{6n}{c} < 16 < \frac{6n}{b} \leq \frac{33}{2}$ . Let  $m = 16$  and  $k = 6$ , then we have done.

**Case 5.**  $t = 3$ .

It holds that  $11 < \frac{4n}{c} < \frac{4n}{b} \leq 12$ , and we infer that  $a < \frac{n}{24}$ . Moreover, we have

$$\frac{5n}{c} \leq 14 < \frac{5n}{b} \leq 15, \quad \frac{6n}{c} \leq 17 < \frac{6n}{b} \leq 18, \quad \frac{7n}{c} \leq 20 < \frac{7n}{b} \leq 21.$$

If  $\gcd(n, 14) = 1$ , let  $m = 14$  and  $k = 5$ , if  $\gcd(n, 17) = 1$ , let  $m = 17$  and  $k = 6$ , if  $\gcd(n, 20) = 1$ , let  $m = 20$  and  $k = 7$ , then  $ma < n$  and  $\text{ind}(S) = 1$ . If none of the three integers is co-prime to  $n$ , then  $n = 5 \times 7 \times 17 = 595 < 1000$ , a contradiction.

**Case 6.**  $t = 4$ .

It holds that  $12 < \frac{4n}{c} < \frac{4n}{b} \leq 13$ , then  $a < \frac{4}{3}(\frac{n}{12} - \frac{n}{13}) < \frac{n}{29}$  and  $15 < \frac{5n}{c} < 16 < \frac{5n}{b} \leq \frac{65}{4}$ . Let  $m = 16$  and  $k = 5$ , we have  $\text{ind}(S) = 1$ .  $\square$

**Lemma 5.5.** *If the assumption is as in Proposition 2.16 and  $k_1 = 4$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Similar to Lemma 5.2, we have  $\frac{n}{c} < \frac{40}{9}$ , then  $6 + t < \frac{3n}{c} < \frac{3n}{b} \leq 7 + t$  for some  $t \in [0, 6]$  or  $13 < \frac{3n}{c} < \frac{3n}{b} \leq \frac{40}{3}$ . We distinguish eight cases.

**Case 1.** It holds that  $13 < \frac{3n}{c} < \frac{3n}{b} \leq \frac{40}{3}$ . We have  $39 < \frac{9n}{c} < \frac{9n}{b} \leq 40$ , which contradicts to the maximality of  $k_1$ .

**Case 2.**  $t = 0$ .

It holds that  $6 < \frac{3n}{c} < \frac{3n}{b} \leq 7$ . We infer that  $a < \frac{n}{10}$  and  $8 < \frac{4n}{c} < 9 < \frac{4n}{b} \leq \frac{28}{3}$ . Let  $m = 9$  and  $k = 4$ , we have  $\text{ind}(S) = 1$ .

**Case 3.**  $t = 1$ .

It holds that  $7 < \frac{3n}{c} < \frac{3n}{b} \leq 8$ , and we infer that  $a < \frac{n}{13}$ . If  $\frac{5n}{c} < 12 < \frac{5n}{b}$ , let  $m = 12$  and  $k = 5$ , then  $\text{ind}(S) = 1$ . We may assume that  $12 < \frac{5n}{c} \leq 13 < \frac{5n}{b} < \frac{40}{3}$ , then  $a < \frac{n}{18}$ . Let  $M = 18$ ,

we have

$$8n > \frac{15n}{2} > 18c > \frac{36n}{5} > 7n > \frac{90n}{13} > 18b > \frac{27n}{4} > 6n,$$

and  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 7n) + (7n - Mb) + (n - Ma) = n$ , then  $\text{ind}(S) = 1$ .

**Case 4.**  $t = 2$ .

It holds that  $8 < \frac{3n}{c} < \frac{3n}{b} \leq 9$ , and we infer that  $a < \frac{n}{18}$ . Since

$$\frac{4n}{c} \leq 11 < \frac{4n}{b}, \frac{5n}{c} \leq 14 < \frac{5n}{b}, \frac{6n}{c} \leq 17 < \frac{6n}{b},$$

we infer that  $n = 7 \times 11 \times 17$ .

If  $18 < \frac{56}{3} < \frac{7n}{c} < 19 < \frac{7n}{b} \leq 20$ , we have  $a < \frac{n}{30}$ , let  $m = 19$  and  $k = 7$ . Then  $\gcd(n, m) = 1$  and  $\text{ind}(S) = 1$ .

If  $19 < \frac{7n}{c} < 20 < \frac{7n}{b} \leq 21$ , we have  $a < \frac{n}{21}$ , let  $m = 20$  and  $k = 7$ . Then  $\gcd(n, m) = 1$  and  $\text{ind}(S) = 1$ .

If  $\frac{56}{3} < \frac{7n}{c} < 19 < 20 < \frac{7n}{b} \leq 21$ , we infer that  $19a > n$ . If  $27c < 10n$ , then  $a < \frac{4}{3}(c - b) < \frac{4}{3}(\frac{10n}{27} - \frac{n}{3}) < \frac{n}{20}$ , a contradiction. So  $\frac{3n}{8} > c > \frac{10n}{27} > \frac{7n}{20} > b > \frac{n}{3}$  and  $\frac{81n}{8} > 27c > 10n > \frac{189n}{20} > 27b > 9n$ ,  $n < 27a < \frac{3n}{2}$ . Let  $M = 27$ , we have  $|Mc|_n > \frac{n}{2}$ ,  $|M(n-b)|_n > \frac{n}{2}$ ,  $|M(n-a)|_n > \frac{n}{2}$ , and  $\text{ind}(S) = 1$ .

**Case 5.**  $t = 3$ .

It holds that  $9 < \frac{3n}{c} < \frac{3n}{b} \leq 10$ , and we infer that  $a < \frac{n}{22}$  and  $\frac{5n}{c} < 16 < \frac{5n}{b}$ . Let  $m = 16$  and  $k = 5$ , then  $\text{ind}(S) = 1$ .

**Case 6.**  $t = 4$ .

It holds that  $10 < \frac{3n}{c} < \frac{3n}{b} \leq 11$ , and we infer that  $a < \frac{n}{27}$ . If  $\frac{5n}{c} < 18 < \frac{5n}{b}$ , let  $m = 18$  and  $k = 5$ , then we have done. If  $\frac{7n}{c} < 24 < \frac{7n}{b}$ , let  $m = 24$  and  $k = 7$ , then we have done. Otherwise, we have  $\frac{5n}{18} < b < c < \frac{7n}{24}$  and  $a < \frac{4}{3}(c - b) < \frac{n}{54}$ . Then  $b > \frac{5n}{18} > \frac{5 \times 54a}{18} = 15a$ , which contradicts to  $s \leq 9$ .

**Case 7.**  $t = 5$ .

It holds that  $11 < \frac{3n}{c} < \frac{3n}{b} \leq 12$ , and we infer that  $a < \frac{n}{33}$  and

$$\frac{4n}{c} \leq 15 < \frac{5n}{b}, \frac{5n}{c} \leq 19 < \frac{5n}{b}, \frac{6n}{c} \leq 23 < \frac{6n}{b}.$$

If  $\gcd(n, 17) = 1$ , let  $m = 17$  and  $k = 4$ , if  $\gcd(n, 21) = 1$ , let  $m = 21$  and  $k = 5$ , if  $\gcd(n, 25) = 1$ , let  $m = 25$  and  $k = 6$ , then  $ma < n$  and  $\text{ind}(S) = 1$ . If none of the three integers is co-prime to  $n$ , then there exists an integer  $m \in [26, 27]$  belongs to  $[\frac{7n}{c}, \frac{7n}{b})$ , let  $k = 7$ , then  $\gcd(m, n) = 1$  and  $\text{ind}(S) = 1$ .

**Case 8.**  $t = 6$ .

It holds that  $12 < \frac{3n}{c} < \frac{3n}{b} \leq 13$ , then we infer that  $a < \frac{n}{39}$  and

$$\frac{4n}{c} \leq 17 < \frac{5n}{b}, \frac{5n}{c} \leq 21 < \frac{5n}{b}, \frac{6n}{c} \leq 25 < \frac{6n}{b}.$$

If  $\gcd(n, 17) = 1$ , let  $m = 17$  and  $k = 4$ , if  $\gcd(n, 21) = 1$ , let  $m = 21$  and  $k = 5$ , if  $\gcd(n, 25) = 1$ , let  $m = 25$  and  $k = 6$ , then  $ma < n$  and  $\text{ind}(S) = 1$ . If none of the three integers is co-prime to  $n$ , then  $n = 5 \times 7 \times 17 = 595 < 1000$ , a contradiction.  $\square$

**Lemma 5.6.** *If the assumption is as in Proposition 2.16 and  $k_1 = 3$ , then  $\text{ind}(S) = 1$ .*

*Proof.* Similar to Lemma 5.2, we have  $\frac{n}{c} < \frac{20}{3}$ , then  $4 + t < \frac{2n}{c} < \frac{2n}{b} \leq 5 + t$  for some  $t \in [0, 8]$  or  $13 < \frac{2n}{c} < \frac{2n}{b} \leq \frac{40}{3}$ . We distinguish ten cases.

**Case 1.** It holds that  $13 < \frac{2n}{c} < \frac{2n}{b} \leq \frac{40}{3}$ . We have  $39 < \frac{6n}{c} < \frac{6n}{b} \leq 40$ , which contradicts to the maximality of  $k_1$ .

**Case 2.**  $t = 0$ .

It holds that  $4 < \frac{2n}{c} < \frac{2n}{b} \leq 5$ . We infer that  $7a < n$ ,  $6 < \frac{3n}{c} \leq 7 < \frac{3n}{b} \leq \frac{15}{2}$ , and  $8 < \frac{4n}{c} < 9 < \frac{4n}{b} \leq 10$ . If  $9a < n$ , let  $m = 9$  and  $k = 4$ , then we have done. If  $9a > n$ , let  $M = 18$ , then  $7n < \frac{36n}{5} < 18b < \frac{54n}{7} < 8n < 18c < 9n$ , and  $18e < 5a < n$ . Then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = 3n$ , and  $\text{ind}(S) = 1$ .

**Case 3.**  $t = 1$ .

It holds that  $5 < \frac{2n}{c} < \frac{2n}{b} \leq 6$ . We infer that  $11a < n$  and  $\frac{3n}{c} < 8 < \frac{3n}{b}$ . Let  $m = 8$  and  $k = 3$ , then  $\text{ind}(S) = 1$ .

**Case 4.**  $t = 2$ .

It holds that  $6 < \frac{2n}{c} < \frac{2n}{b} \leq 7$ . We infer that  $15a < n$  and

$$\frac{3n}{c} < 10 < \frac{3n}{b}, \frac{4n}{c} < 13 < \frac{4n}{b},$$

thus  $\gcd(n, 10) > 1, \gcd(n, 13) > 1$ .

*Subcase 4.1.*  $\frac{5n}{c} < 16 < \frac{5n}{b} \leq \frac{35}{2}$ . If  $16a < n$ , let  $m = 16$  and  $k = 5$ , then we have done. If  $16a > n$ , let  $M = 18$ , we have  $5n < 18b < 18c < 6n$  and  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = 3n$ . Then  $\text{ind}(S) = 1$ .

*Subcase 4.2.*  $16 < \frac{5n}{c} \leq 17 < \frac{5n}{b} \leq \frac{35}{2}$ . We infer that  $28a < n$ . Then  $\gcd(n, 17) > 1$  and  $n = 5 \times 13 \times 17$ . Let  $M = 27$ , we have

$$9n > \frac{135n}{16} > 27c > \frac{81n}{10} > 8n = \frac{136n}{17} > \frac{135n}{17} > 27b > \frac{54n}{7} > 7n,$$

then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = 3n$  and  $\text{ind}(S) = 1$ .

**Case 5.**  $t = 3$ .

It holds that  $7 < \frac{2n}{c} < \frac{2n}{b} \leq 8$ . We infer that  $21a < n$  and

$$\frac{3n}{c} < 11 < \frac{3n}{b}, \frac{4n}{c} < 15 < \frac{4n}{b},$$

thus  $\gcd(n, 11) > 1, \gcd(n, 15) > 1$ .

*Subcase 5.1.*  $\frac{5n}{c} < 18 < \frac{5n}{b} \leq 20$ . Let  $m = 18$  and  $k = 5$ , then we have done.

*Subcase 5.2.*  $18 < \frac{5n}{c} \leq 19 < \frac{5n}{b} \leq 20$ . We infer that  $27a < n$  and thus  $\gcd(n, 19) > 1$ . In further,  $\frac{7n}{c} \leq \frac{77}{3} < 26 < \frac{133}{5} < \frac{7n}{b}$ , let  $m = 26$  and  $k = 7$ , then  $\gcd(n, 26) = 1$  and  $\text{ind}(S) = 1$ .

**Case 6.**  $t = 4$ .

It holds that  $8 < \frac{2n}{c} < \frac{2n}{b} \leq 9$ . We infer that  $27a < n$  and

$$\frac{3n}{c} < 13 < \frac{3n}{b}, \frac{4n}{c} < 17 < \frac{4n}{b},$$

thus  $\gcd(n, 13) > 1, \gcd(n, 17) > 1$ .

*Subcase 6.1.*  $\frac{5n}{c} \leq 21 < 22 < \frac{5n}{b}$ . Let  $k = 5$  and  $m \in [21, 22]$  such that  $\gcd(n, m) = 1$ , then we have done.

*Subcase 6.2.*  $21 < \frac{5n}{c} \leq 22 < \frac{5n}{b} \leq \frac{45}{2}$ . We infer that  $a < \frac{4n}{189}$  and thus  $\frac{b}{a} > \frac{2}{9} \times \frac{189}{4} = \frac{21}{2} > 10$ , which contradicts to  $s \leq 9$ .

*Subcase 6.3.*  $20 < \frac{5n}{c} \leq 21 < \frac{5n}{b} \leq 22$ . We infer that  $33a < n$ ,  $7|n$  and thus  $\gcd(n, 33) = 1$ . Let  $M = 33$ , we have  $\frac{15n}{2} < 33b < \frac{99n}{13}$ ,  $\frac{n}{2} < 33a < n$  (otherwise,  $66a < n < \frac{22b}{5}$ , thus  $b > 15a$ , a contradiction) and  $33e < 9a < n$ . Then  $|Me|_n < \frac{n}{2}$ ,  $|M(n-b)|_n < \frac{n}{2}$ ,  $|M(n-a)|_n < \frac{n}{2}$ , and  $\text{ind}(S) = 1$ .

**Case 7.**  $t = 5$ .

It holds that  $9 < \frac{2n}{c} < \frac{2n}{b} \leq 10$ . We infer that  $33a < n$ . If  $\frac{5n}{c} < 24 < \frac{5n}{b}$ . Let  $k = 5$  and  $m = 24$ , then  $\gcd(n, m) = 1$  and  $\text{ind}(S) = 1$ . Otherwise, assume that  $\frac{45}{2} < \frac{5n}{c} \leq 23 < \frac{5n}{b} < 24$ . We infer that  $54a < n$  and  $\frac{b}{a} > \frac{5}{24} \times 54 = \frac{90}{8} > 10$ , which contradicts to  $s \leq 9$ .

**Case 8.**  $t = 6$ .

It holds that  $10 < \frac{2n}{c} < \frac{2n}{b} \leq 11$ . We infer that  $41a < n$  and  $\frac{3n}{c} < 16 < \frac{3n}{b}$ . Let  $k = 3$  and  $m = 16$ , then  $\gcd(n, m) = 1$  and  $\text{ind}(S) = 1$ .

**Case 9.**  $t = 7$ .

It holds that  $11 < \frac{2n}{c} < \frac{2n}{b} \leq 12$ . We infer that  $49a < n$  and

$$\frac{3n}{c} < 17 < \frac{3n}{b}, \frac{4n}{c} < 23 < \frac{4n}{b},$$

thus  $\gcd(n, 17) > 1, \gcd(n, 23) > 1$ .

*Subcase 9.1.*  $\frac{5n}{c} \leq 28 < 29 < \frac{5n}{b}$ . Let  $k = 5$  and  $m \in [28, 29]$  such that  $\gcd(n, m) = 1$ , then we have done.

*Subcase 9.2.*  $28 < \frac{5n}{c} \leq 29 < \frac{5n}{b} \leq 30$ . We infer that  $a < \frac{n}{63}$  and thus  $\frac{b}{a} > \frac{21}{2} > 10$ , which contradicts to  $s \leq 9$ .

*Subcase 9.3.*  $\frac{55}{2} < \frac{5n}{c} \leq 28 < \frac{5n}{b} \leq 29$ . Similar to *Subcase 9.2.*, we get a contradiction.

**Case 10.**  $t = 8$ .

It holds that  $12 < \frac{2n}{c} < \frac{2n}{b} \leq 13$ , we infer that  $a < \frac{n}{58}$ . If  $\frac{5n}{c} < 32 < \frac{5n}{b}$ , let  $m = 32$  and  $k = 5$ , then we have done. Otherwise, we have  $30 < \frac{5n}{c} < \frac{5n}{b} < 32$  and  $a < \frac{4}{3}(a-e) = \frac{4}{3}(c-b) < \frac{4}{3}(\frac{n}{6} - \frac{5n}{32}) = \frac{n}{72}$ . Then  $b > \frac{5n}{32} > \frac{5 \times 72a}{32} = \frac{45a}{4} > 10a$ , which contradicts to  $s \leq 9$ .  $\square$

**Lemma 5.7.** Suppose that  $p_3 = 5$ ,  $p_1 \geq 13$  and  $6a < n$ . If  $e = 2p_2$ , then  $p_1 \geq 23$  and  $n > 57e$ . If  $a \leq p_1$ , then  $n > 85e$ .

*Proof.* This result can be checked directly.  $\square$

**Lemma 5.8.** Let the assumption be as in Proposition 2.16. If  $k_1 = 2$ ,  $4 < \frac{2n}{c} \leq 5 < \frac{2n}{b} \leq 6$  and  $5|n$ , then  $\text{ind}(S) = 1$ .

*Proof.* If  $n < 6a$ ,  $me < 2a < n$ ,  $2n < 6b < 6c < 3n$ , let  $M = 6$ . Then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n \geq Me + (Mc - 2n) + (3n - Mb) + (2n - Ma) = 3n$ , and  $\text{ind}(S) = 1$ . Next we assume that  $6a < n$  and distinguish three cases.

**Case 1.**  $7 < \frac{3n}{c} \leq 8 < \frac{3n}{b} \leq 9$ .

If  $8a < n$ , let  $m = 8$  and  $k = 3$ , we have  $\text{ind}(S) = 1$ . If  $8a > n$ , let  $M = 9$ , we have  $3n < 9b < 9c < 4n$ ,  $9e < 3a < n$  and  $9a < 2n$ , so  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 3n) + (4n - Mb) + (2n - Ma) = 3n$ . Then  $\text{ind}(S) = 1$ .

**Case 2.**  $6 < \frac{3n}{c} \leq 7 < \frac{3n}{b} \leq 8$ .

If  $8a > n$ , let  $M = 8$ , we have  $3n < 8b < 8c < 4n$ ,  $8e < 2a < n$  and  $8a < 2n$ , so  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 3n) + (4n - Mb) + (2n - Ma) = 3n$ . Then  $\text{ind}(S) = 1$ .

Next assume that  $8a < n$ , then  $7|n$ ,  $\gcd(n, 11) = \gcd(n, 13) = 1$  and  $\frac{11n}{2} > 11c > \frac{33n}{7} > \frac{22n}{5} > 11b > \frac{33n}{8}$ .

If  $11c < 5n$ , let  $M = 12$ , we have  $\frac{60n}{11} > 12c > \frac{36n}{7} > \frac{24n}{5} > 12b > \frac{9n}{2}$  and  $12e < 3a < \frac{n}{2}$ . Then  $|Me|_n < \frac{n}{2}$ ,  $|Mc|_n < \frac{n}{2}$ ,  $|M(n-b)|_n < \frac{n}{2}$ , and  $\text{ind}(S) = 1$ .

If  $11c > 5n$  and  $11a < n$ , let  $M = 11$ , then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 5n) + (5n - Mb) + (n - Ma) = n$ , and  $\text{ind}(S) = 1$ .

Let  $11c > 5n$  and  $11a > n$ . If  $18b > 7n$ , let  $M = 9$ , we have  $\frac{9n}{2} > 9c > \frac{45n}{11} > \frac{18n}{5} > 9b > \frac{7n}{2}$ , then  $|Me|_n < \frac{n}{2}$ ,  $|Mc|_n < \frac{n}{2}$ ,  $|M(n-b)|_n < \frac{n}{2}$ , and  $\text{ind}(S) = 1$ . If  $18b < 7n$ , let  $M = 18$ , we have  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 8n) + (7n - Mb) + (2n - Ma) = n$ , and  $\text{ind}(S) = 1$ .

If  $11c > 5n$  and  $11a > n$ , let  $M = 9$ , then  $\frac{65n}{11} > 13c > \frac{39n}{7} > \frac{26n}{5} > 9b > \frac{39n}{8}$ . We infer that  $|Mc|_n < \frac{n}{2}$ ,  $|M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 4n) + (5n - Mb) + (2n - Ma) = 3n$ , and  $\text{ind}(S) = 1$ .

**Case 3.**  $6 < \frac{3n}{c} \leq 7 < 8 < \frac{3n}{b} \leq 9$ .

If  $\frac{27}{4} < \frac{3n}{c} \leq 7 < 8 < \frac{3n}{b} \leq 9$ . If  $8a < n$ , let  $m = 8$  and  $k = 3$ , then  $\text{ind}(S) = 1$ . If  $8a > n$ , let  $M = 9$ , then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 3n) + (4n - Mb) + (2n - Ma) = 3n$ , and  $\text{ind}(S) = 1$ .

Next assume that  $6 < \frac{3n}{c} < \frac{27}{4} < 7 < 8 < \frac{3n}{b} \leq 9$ .

If  $p_1 = 5$ , then  $n > 200e$ . We only need repeat the proof of Case 3 of Lemma 3.10 in [11]. Then  $p_3 = 5$ .

*Subcase 3.1.*  $\gcd(n, 7) = \gcd(n, 11) = 1$ .

We infer that  $7a > n$  and  $n \geq 85p_1$ .

If  $11b < 4n$  and  $11c > 5n$ , we have  $|11e|_n + |11c|_n + |11(n-b)|_n + |11(n-a)|_n = n$  and thus  $\text{ind}(S) = 1$ .

If  $11b > 4n$  and  $11c < 5n$ , we have  $|11e|_n + |11c|_n + |11(n-b)|_n + |11(n-a)|_n = 3n$  and thus  $\text{ind}(S) = 1$ .

If  $11b < 4n$  and  $11c < 5n$ , then  $\frac{n}{7} < a = c - b + e \leq \frac{5n - ep_3}{11} - \frac{n+e}{3} + e \leq \frac{4n+7e}{33}$ , so  $n < 10e$ , or  $\frac{n}{7} < a = \frac{10}{9}(c-b) < \frac{40n}{297}$ , either of them implies a contradiction.

If  $11b > 4n$  and  $11c > 5n$ , then  $\frac{n}{7} < a = c - b + e \leq \frac{n-e}{2} - \frac{4n+p_3e}{11} + e \leq \frac{3n+e}{22}$ , so  $n < 7e$ , or  $\frac{n}{7} < a = c - b + e \leq \frac{n-p_1p_3}{2} - \frac{4n+p_1}{11} + e \leq \frac{3n+12e}{22}$ , so  $n < 84e$ , or  $\frac{n}{7} < a = c - b + e \leq \frac{n-p_1p_3}{2} - \frac{4n+p_1}{11} + e \leq \frac{3n+8e}{22}$ , so  $n < 56e$ . By Lemma 5.7, each of above implies a contradiction.

*Subcase 3.2.*  $11|n$ . We infer that  $8a > n$ ,  $e = 11$ , and  $n \geq 95e$ .



The proof is similar to *Subcase 3.1*.

*Subcase 3.3.*  $7|n$ . We infer that  $8a > n$ ,  $e = 7$ , and  $n \geq 145e$ .

The proof is similar to *Subcase 3.1*. □

**Lemma 5.9.** *If the assumption is as in Proposition 2.16 and  $k_1 = 2$ , then  $\text{ind}(S) = 1$ .*

*Proof.* By Lemma 2.5, it holds that  $2 + t < \frac{n}{c} < \frac{n}{b} \leq 3 + t$  for some  $t \in [0, 5]$ . We distinguish six cases.

**Case 1.**  $t = 0$ . Then  $2 < \frac{n}{c} < \frac{n}{b} \leq 3$  and  $4 < \frac{2n}{c} \leq 5 < \frac{2n}{b} \leq 6$ .

Similar to Lemma 5.7, we infer that  $6a < n$  and hence  $5|n$ , by Lemma 5.8,  $\text{ind}(S) = 1$ .

**Case 2.**  $t = 1$ . Then  $3 < \frac{n}{c} < \frac{n}{b} \leq 4$ . We infer that  $a < \frac{n}{9}$  and  $6 < \frac{2n}{c} \leq 7 < \frac{2n}{b} \leq 8$ . Thus  $7|n$ .

If  $9 < \frac{3n}{c} \leq 10 < 11 < \frac{3n}{b} \leq 12$ , let  $m \in [10, 11]$  (since  $n$  can't be  $5 \times 7 \times 11 = 385$ ) such that  $\gcd(n, m) = 1$ . If  $ma < n$ , let  $k = 3$ , then  $\text{ind}(S) = 1$ . If  $ma > n$ , let  $M = 12$ , we have  $Me < 4a < n$ ,  $n < Ma < 2n$  and  $3n < Mb < Mc < 4n$ , then  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 3n) + (4n - Mb) + (2n - Ma) = 3n$ , and  $\text{ind}(S) = 1$ .

If  $9 < \frac{3n}{c} \leq 10 < \frac{3n}{b} \leq 11$ , we infer that  $a < \frac{n}{12}$  and  $5|n$ . If  $12 < \frac{4n}{c} \leq 13 < \frac{4n}{b} \leq \frac{44}{3}$ , we infer that  $13a > n$ . Otherwise, let  $m = 13$  and  $k = 4$ , we have  $\gcd(n, 13) = 1$  (otherwise,  $n = 5 \times 7 \times 13 = 455 < 1000$ ), then  $\text{ind}(S) = 1$ . Let  $M = 22$ , it is easy to check that  $\gcd(n, M) = 1$ . If  $Mc < 7n$ , we have  $Me < n$ ,  $n < Ma < 2n$  and  $6n < Mb < Mc < 7n$ , and  $|Me|_n + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n = Me + (Mc - 6n) + (7n - Mb) + (2n - Ma) = 3n$ , thus  $\text{ind}(S) = 1$ . If  $Mc > 7n$ , we have  $|Me|_n < \frac{n}{2}$ ,  $|Mc|_n < \frac{n}{2}$  and  $|M(n-a)|_n < \frac{n}{2}$ , then  $\text{ind}(S) = 1$ .

If  $10 < \frac{3n}{c} \leq 11 < \frac{3n}{b} \leq 12$ , we infer that  $a < \frac{n}{15}$  and  $11|n$ . If  $\frac{4n}{c} \leq 15 < \frac{4n}{b}$ , then  $\gcd(15, n) = 1$  and  $15a < n$ . Let  $m = 15$  and  $k = 4$ , we have  $\text{ind}(S) = 1$ . So  $13 < \frac{40}{3} < \frac{4n}{c} \leq 14 < \frac{4n}{b} \leq 15$ , and we infer that  $a < \frac{n}{22}$ . Let  $M = 25$ , we have  $\gcd(n, M) = 1$  and

$$6n + \frac{2n}{3} = \frac{100n}{15} < Mb < \frac{75n}{11} = 6n + \frac{9n}{11} < 7n < 7n + \frac{n}{7} = \frac{50n}{7} < Mc < \frac{15n}{2} = 7n + \frac{n}{2}.$$

Then  $|Me|_n < \frac{n}{2}$ ,  $|Mc|_n < \frac{n}{2}$  and  $|M(n-b)|_n < \frac{n}{2}$ , thus  $\text{ind}(S) = 1$ .

**Case 3.**  $t = 2$ . Then  $4 < \frac{n}{c} < \frac{n}{b} \leq 5$ . We infer that  $a < \frac{n}{15}$  and  $8 < \frac{n}{c} < 9 < \frac{n}{b} \leq 10$ . Let  $m = 9$  and  $k = 2$ , then  $\text{ind}(S) = 1$ .

**Case 4.**  $t = 3$ . Then  $5 < \frac{n}{c} < \frac{n}{b} \leq 6$ ,  $10 < \frac{2n}{c} \leq 11 < \frac{2n}{b} \leq 12$  and  $a < \frac{n}{22}$ . If  $\gcd(n, 11) = 1$ , let  $m = 11$  and  $k = 2$ , we have  $\text{ind}(S) = 1$ . If  $\frac{3n}{c} < 16 < \frac{3n}{b}$ , let  $m = 16$  and  $k = 3$ , then  $\text{ind}(S) = 1$ . Otherwise, we have  $a < \frac{4}{3}(a - e) = \frac{4}{3}(c - b) < \frac{4}{3}(\frac{3n}{16} - \frac{n}{6}) = \frac{n}{36}$ , and  $16 < \frac{3n}{c} \leq 17 < \frac{3n}{b} < 18$ . Similarly,  $\gcd(n, 17) > 1$  and  $27 < \frac{5n}{c} < \frac{5n}{b} \leq 30$ .

If  $27 < \frac{5n}{c} \leq 28 < \frac{5n}{b} \leq 29$ , we have  $a < \frac{4 \times 2 \times 5n}{3 \times 27 \times 29}$ , and  $b > \frac{5n}{29} > \frac{5}{29} \frac{3 \times 27 \times 29a}{4 \times 2 \times 5} = \frac{81a}{8} > 10a$ , which contradicts to  $s \leq 9$ .

If  $28 < \frac{5n}{c} \leq 29 < \frac{5n}{b} \leq 30$ , we have  $a < \frac{4 \times 2 \times 5n}{3 \times 28 \times 30}$ , and  $b > \frac{n}{6} > \frac{1}{6} \frac{3 \times 28 \times 30a}{4 \times 2 \times 5} = \frac{21a}{2} > 10a$ , which contradicts to  $s \leq 9$ .

If  $27 < \frac{5n}{c} \leq 28 < 29 < \frac{5n}{b} \leq 30$ , then there exists  $m$  between  $\frac{5n}{c}$  and  $\frac{5n}{b}$  ( $m = 28$  or  $m = 29$ ) such that  $\gcd(n, m) = 1$ . Let  $k = 5$ , we have  $\text{ind}(S) = 1$ .

**Case 5.**  $t = 4$ . Then  $6 < \frac{n}{c} < \frac{n}{b} \leq 7$ . We infer that  $12 < \frac{2n}{c} \leq 13 < \frac{2n}{b} \leq 14$ ,  $a < \frac{n}{31}$  and thus  $\gcd(n, 13) > 1$ .

If  $18 < \frac{3n}{c} \leq 19 < 20 < \frac{3n}{b} \leq 21$ , we infer that  $n = 5 \times 13 \times 19$ . If  $\frac{5n}{c} \leq 31 < \frac{5n}{b}$ , let  $m = 31$  and  $k = 5$ , we have  $\gcd(n, m) = 1$ , then  $\text{ind}(S) = 1$ . Otherwise,  $31 < \frac{5n}{c} < \frac{5n}{b} \leq 35$ , hence  $a < \frac{4}{3} \times (\frac{5n}{31} - \frac{5n}{35}) = \frac{n}{40}$ . Then, for any integer  $m$  between  $\frac{5n}{c}$  and  $\frac{5n}{b}$ , we have  $\gcd(n, m) = 1$  and  $ma < n$ , hence  $\text{ind}(S) = 1$ .

If  $18 < \frac{3n}{c} \leq 19 < \frac{3n}{b} \leq 20$ , we infer that  $a < \frac{n}{45}$ . If  $\frac{5n}{c} < 32 < \frac{5n}{b}$ , let  $m = 32$  and  $k = 5$ , then we have done. Otherwise,  $30 < \frac{5n}{c} \leq \frac{95}{3} < \frac{5n}{b} < 32$ , thus we infer that  $a < \frac{n}{72}$  and  $\frac{b}{a} > \frac{5 \times 72}{32} = \frac{45}{4} > 10$ , which contradicts to  $s \leq 9$ .

If  $19 < \frac{3n}{c} \leq 20 < \frac{3n}{b} \leq 21$ , we infer that  $a < \frac{n}{49}$  and  $\gcd(n, 20) > 1$ . If  $27 < \frac{4n}{b}$ , then  $\frac{4n}{c} \leq \frac{80}{3} < 27 < \frac{4n}{b}$ , let  $m = 27$  and  $k = 4$ , we have  $\text{ind}(S) = 1$ . Otherwise,  $\frac{76}{3} \frac{4n}{c} < \frac{4n}{b} < 27$ , we infer that  $a < \frac{4 \times 5n}{19 \times 81}$ , and  $\frac{b}{a} > \frac{4}{27} \times \frac{19 \times 81}{20} = \frac{57}{5} > 10$ , which contradicts to  $s \leq 9$ .

**Case 6.**  $t = 5$ . Then  $7 < \frac{n}{c} < \frac{n}{b} \leq 8$ . We infer that  $14 < \frac{2n}{c} \leq 15 < \frac{2n}{b} \leq 16$ ,  $a < \frac{n}{42}$  and thus  $\gcd(n, 15) > 1$ .

If  $21 < \frac{3n}{c} \leq 22 < 23 < \frac{3n}{b} \leq 24$ , we infer that  $n = 5 \times 11 \times 23$ . If 29 or 31 belongs to  $[\frac{4n}{c}, \frac{4n}{b})$ , let it be  $m$  and  $k = 4$ , we have  $\gcd(n, m) = 1$ , then  $\text{ind}(S) = 1$ . Otherwise,  $29 < \frac{4n}{c} \leq 30 < \frac{4n}{b} \leq 31$ , hence  $a < \frac{4}{3} \times (\frac{4n}{29} - \frac{4n}{31}) = \frac{32n}{3 \times 29 \times 31}$ , and  $\frac{b}{a} > \frac{4}{31} \times \frac{3 \times 29 \times 31}{32} = \frac{87}{8} > 10$ , which contradicts to  $s \leq 9$ .

If  $21 < \frac{3n}{c} \leq 22 < \frac{3n}{b} \leq 23$ , we infer that  $a < \frac{n}{60}$ . If  $\frac{5n}{c} < 36 < \frac{5n}{b}$ , let  $m = 36$  and  $k = 5$ , then we have done. Otherwise,  $36 < \frac{5n}{c} \leq 37 < \frac{5n}{b} \leq \frac{115}{3}$ , thus we infer that  $a < \frac{7n}{23 \times 27}$  and  $\frac{b}{a} > \frac{23 \times 27}{7} \times \frac{3}{23} = \frac{81}{3} > 10$ , which contradicts to  $s \leq 9$ .

If  $22 < \frac{3n}{c} \leq 23 < \frac{3n}{b} \leq 24$ , we infer that  $a < \frac{n}{66}$  and  $\gcd(n, 23) > 1$ . If  $\frac{5n}{c} \leq 37$ , then  $\frac{5n}{c} \leq 37 < 38 < \frac{115}{3} < \frac{5n}{b} \leq 40$ . There exists  $m \in [37, 38]$  such that  $\gcd(m, n) = 1$ , let  $k = 5$ , then  $\text{ind}(S) = 1$ . Similarly,  $\frac{5n}{c} \leq 38 < 39 < \frac{5n}{b} \leq 40$  implies  $\text{ind}(S) = 1$ . If  $37 < \frac{5n}{c} \leq 38 < \frac{5n}{b} \leq 39$  or  $38 < \frac{5n}{c} \leq 39 < \frac{5n}{b} \leq 40$ , we infer that  $b > 10a$ , a contradiction.  $\square$

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